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Exact observability, square functions and spectral theory ☆

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Abstract

In the first part of this article we introduce the notion of a backward–forward conditioning (BFC) system that generalises the notion of zero-class admissibility introduced in Xu et al. (2008) [30]. We can show that unless the spectrum contains a half-plane, the BFC property occurs only in situations where the underlying semigroup extends to a group. In a second part we present a sufficient condition for exact and final state observability in Banach spaces that is designed for infinite-dimensional output spaces and general strongly continuous semigroups. To obtain this we make use of certain weighted square function estimates. Specialising to the Hilbert space situation we obtain a result for contraction semigroups without an analyticity condition on the semigroup.

Keywords: Exact observability; Final state observability; Admissibility; Spectral theory of semigroups and groups; H^∞ functional calculus; Square function estimates

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1. Introduction

In this article we study exact observability of linear systems (A, C) of the form

$$\begin{cases} x'(t) + Ax(t) = 0, \\ x(0) = x_0, \\ y(0) = Cx(t). \end{cases}$$

We suppose throughout this article that $-A$ is the generator of a strongly continuous semigroup $T(t)_{t\geqslant 0}$ on a Banach space X. For details on semigroup theory used frequently in this article we refer to e.g. to the textbooks [6,8,24]. Since we deal with unbounded operators in general, we will note $\mathcal{D}(A)$ the domain of A and $\mathcal{R}(A)$ its range.

Let Y be another Banach space and suppose that the observation operator $C : \mathcal{D}(A) \to Y$ is bounded and linear when $\mathcal{D}(A)$ is endowed with the graph norm $\|x\|_{\mathcal{D}(A)} = \|x\| + \|Ax\|$. Here we denote by $\|\cdot\|$ the norm of X. Since the observation operator C is generally unbounded, the concept of admissibility is introduced. It means that the output y of the system (usually measured in L^2 norm) depends continuously on the initial value x_0.

Definition 1.1. We say that C is L^2-admissible in time $\tau > 0$ (for A or for $T(t)_{t\geqslant 0}$) if there exists a constant $M(\tau) > 0$ such that

$$\sup_{x\in\mathcal{D}(A),\|x\|=1} \int_0^\tau \|CT(t)x\|_Y^2 \, dt =: M(\tau)^2 < \infty.$$

Definition 1.2. We say that C is exactly L^2-observable for A (or for $T(t)$) in time $\eta > 0$ if there exists a constant $m(\eta) > 0$ such that

$$\inf_{x\in\mathcal{D}(A),\|x\|=1} \int_0^\eta \|CT(t)x\|_Y^2 \, dt =: m(\eta)^2 > 0.$$

For more information on the notion of admissible observation (or control) operators we refer the reader to the overview article [9] or, both for admissibility and observability issues to the books [29,15,16] and references therein. We summarise some well-known facts and notations.

When there is no risk of confusion, ‘admissible’ means L^2 admissible in *some* finite time $\tau > 0$ and ‘exact observable’ means exactly L^2-observable for A in *some* finite time $\eta > 0$. We say that C is infinite-time admissible if $M(\infty) < \infty$ and exactly observable in infinite time if $m(\infty) > 0$.

Note that it may be possible that exact observability does not hold for the system (A, C) but there exists a positive time η and a constant $m(\eta) > 0$ such that

$$\int_0^\eta \|CT(t)x\|_Y^2 \, dt \geqslant m(\eta)^2 \|T(\eta)x\|^2.$$

Such estimate is called the *final state observability* and corresponds for the dual problem to null controllability. See, e.g. [29].

Finite-time admissibility does not depend on the choice of $\tau > 0$. Nevertheless it turns out to be useful to study the (clearly non-decreasing) functions $t \mapsto m(t)$ and $t \mapsto M(t)$. In the 'dual' situation of a control operator B the quantity $m(\eta)^{-1}$ is often referred to as *control cost* of a system. We refer e.g. to [20,26,27] and references therein for more details.

Independence of the time $\tau > 0$ of the notion of admissibility means that a lack of admissibility expresses either by $M(\tau) = \infty$ for all $\tau > 0$ or by $M(\tau) < \infty$ for finite τ while $M(\infty) = \infty$. On the other hand, a lack of exact observability expresses by $m(\eta) = 0$ for $0 < \eta < \eta_0$ for $\eta_0 \in (0, \infty]$. We remark that Example 2.3 below satisfies $m(\eta) = 0$ for $0 < \eta < 2$, $m(2) = 1$ while $m(\eta) \to +\infty$ for $\eta \to +\infty$.

Since most parabolic equations like for example the heat equation are not exactly observable unless very special observations are chosen whereas exact observability appears frequently for hyperbolic systems such as the wave equation, it appears natural to study necessary spectral conditions of the generator $-A$ that make exact observability possible or impossible. In this direction we extend and complete former results of [30]. We introduce the notion of backward–forward conditioning BFC-systems. These are admissible and exactly observable systems for which $M(\eta) < m(\tau)$ for some $\eta < \tau$. We analyse spectral properties of the generator $-A$ of the semigroup of such systems. In particular, we prove that the approximate point spectrum of A is contained in a vertical strip. Therefore, the boundary of the spectrum is also contained in a strip. We prove in addition that if (A, C) is an admissible BFC-system such that the spectrum of A does not contain a half-plane then the semigroup actually extends to a group. Note that every bounded group with an admissible operator C is a BFC-system. Since (BFC) is a frequent property our results show that exact observability is considerably rare outside the group context.

For exact observability, resolvent conditions like the "Hautus test" (see [25]) provide necessary conditions. In some situations, such as norm-continuous semigroups or special diagonal operators generators they are also sufficient (see e.g. [25,11]), and the same is true for certain group generators, see e.g. [19,12,5]. However, for general semigroups little is known so far. A second part of this paper is therefore devoted to new sufficient conditions for exact or final state observability. Under an assumption of lower square function estimate we prove that a condition of type

$$\left\| CA^{-\alpha}x \right\|_Y \geqslant \delta \|x\|,$$

implies exact observability. Here $\alpha \in (0, 1)$ and δ is a positive constant. If for some $\tau > 0$

$$\left\| CA^{-\alpha}x \right\|_Y \geqslant \delta \left\| T(\tau)x \right\|,$$

then we obtain final state observability.

In order to state and prove our criteria we make heavy use of square function estimate of type

$$\|x\|^2 \leqslant K^2 \int_0^\infty \left\| (tA)^{-\beta}\left(T\left(2t^{2\beta}\right) - T\left(t^{2\beta}\right)\right)x \right\|^2 \frac{dt}{t},$$

where K is a positive constant and $\beta \in (0, 1)$. In the case where $\beta = \frac{1}{2}$, this corresponds to the lower square function estimate

$$\|x\|^2 \leqslant K^2 \int_0^\infty \|\varphi(tA)x\|^2 \frac{dt}{t},$$

where $\varphi(z) := z^{-\frac{1}{2}}(e^{-2z} - e^{-z})$. On Hilbert spaces, it is well known that square function estimates are related to the holomorphic functional calculus of the operator A. See [18] and [3]. However this theory requires the fact that the function φ is bounded holomorphic on a sector which is strictly larger than the spectrum $\sigma(A)$ of A. In one of our results we prove a square function estimate for $\varphi(z) := z^{-\frac{1}{2}}(e^{-2z} - e^{-z})$ and any accretive operator A on a Hilbert space. Note that in this case $\sigma(A)$ is contained in the closed half-plane and φ is not bounded on any strictly larger sector. As a consequence, we obtain without additional assumptions that if $-A$ is the generator of a contraction semigroup on a Hilbert space and $C : \mathcal{D}(A) \to Y$ is such that $\|CA^{-\frac{1}{2}}x\|_Y \geqslant \delta\|x\|$, then (A, C) is exactly observable. Similarly, if $\|CA^{-\frac{1}{2}}x\|_Y \geqslant \delta\|T(\tau)x\|$, then the system is final state observable.

As we will explain later our criterion applies to any bounded analytic semigroup on a Hilbert space whose generator admits a bounded H^∞ functional calculus. However, the first part of this paper reveals this to be impossible for a large class of systems unless A is bounded. One important aspect of our criteria is therefore to avoid making use of analyticity assumption of the semigroup. Our ideas of relating square function estimates to the observability problem may be helpful in other circumstances where the semigroup is not analytic.

2. BFC-systems

Let X and Y be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_Y$, respectively. Throughout this section, $(T(t))_{t\geqslant 0}$ is a strongly continuous semigroup on X whose generator is denoted by $-A$.

Definition 2.1. An admissible observation operator C for A is called *zero-class admissible* if $\lim_{\tau\to 0+} M(\tau) = 0$.

It is an obvious fact that every bounded operator $C : X \to Y$ is zero-class admissible. Here $(T(t))_{t\geqslant 0}$ is merely a strongly continuous semigroup on X. By a Laplace transform argument it can be seen that the infinite-time admissibility of C implies the so-called Weiss condition

$$\left\|\sqrt{\mathrm{Re}(\lambda)}CR(\lambda, -A)\right\| \leqslant M,$$

where we denote by $-A$ the generator of the semigroup and $R(\lambda, -A) = (\lambda I + A)^{-1}$ is the associated resolvent. It is known that this condition is not characterising (see, e.g. [9]). However, on Hilbert spaces, for all $\alpha \in [0, \frac{1}{2})$, A^α is zero-class admissible. Consequently, if C is bounded on such a fractional domain space $\mathcal{D}(A^\alpha)$, C is zero-class admissible. Even more is true: as pointed out in [10], Zwart's condition

$$\left\|\bigl(1 + \log_+\bigl(\mathrm{Re}(\lambda)\bigr)\bigr)^\alpha \sqrt{\mathrm{Re}(\lambda)}CR(\lambda, -A)\right\| \leqslant M \tag{2.1}$$

for all $\lambda \in \mathbb{C}_+$ and some $\alpha > \frac{1}{2}$ implies not only finite-time admissibility but directly zero-class admissibility. This follows from Zwart's original dyadic decomposition argument that we sketch here:

$$\begin{aligned}\int_0^\tau \|CT(t)x\|_Y^2\,dt &= \sum_{n\geqslant 0}\int_{2^{-n-1}\tau}^{2^{-n}\tau} \|CT(t)x\|_Y^2\,dt\\ &\sim \sum_{n\geqslant 0}\int_{2^{-n-1}\tau}^{2^{-n}\tau} \left\|t\frac{2^n}{\tau}e^{-t\frac{2^n}{\tau}}CT(t)x\right\|_Y^2 dt\\ &\leqslant \sum_{n\geqslant 0}\int_0^\infty \left\|t\frac{2^n}{\tau}e^{-t\frac{2^n}{\tau}}CT(t)x\right\|_Y^2 dt.\end{aligned}$$

Here and in what follows we use the notation $\sim$ and $\lesssim$ to denote equality and inequality up to a constant. Now we observe that the semigroup $e^{-t\frac{2^n}{\tau}}T(t)$ is generated by $-A-\frac{2^n}{\tau}$ and hence by the Laplace transform we have

$$\begin{aligned}\int_0^\tau \|CT(t)x\|_Y^2\,dt &\lesssim \sum_{n\geqslant 0}\int_{\mathbb{R}} \left\|\frac{2^n}{\tau}CR\left(is+\frac{2^n}{\tau},-A\right)^2 x\right\|_Y^2 ds\\ &\lesssim \sum_{n\geqslant 0}\frac{2^n}{\tau(1+\log_+(\frac{2^n}{\tau}))^{2\alpha}}\int_{\mathbb{R}}\left\|R\left(is+\frac{2^n}{\tau},-A\right)x\right\|^2 ds\\ &\lesssim \sum_{n\geqslant 0}\frac{2^n}{\tau(1+\log_+(\frac{2^n}{\tau}))^{2\alpha}}\int_0^\infty \left\|e^{-s\frac{2^n}{\tau}}T(s)x\right\|^2 ds\\ &\lesssim \|x\|^2\sum_{n\geqslant 0}\frac{2^n}{\tau(1+\log_+(\frac{2^n}{\tau}))^{2\alpha}}\int_0^\infty e^{-s2^{n+1}/\tau}\,ds\\ &= \|x\|^2\sum_{n\geqslant 0}\frac{1}{2(1+\log_+(\frac{2^n}{\tau}))^{2\alpha}} =: c_\tau\|x\|^2.\end{aligned}$$

Notice that the first estimate follows from Plancherel's theorem for L^2-functions with values in Y and the third one for L^2-functions with values in X. Both X and Y are assumed to be Hilbert spaces. The fourth estimate follows from the boundedness of the semigroup. It is easy to see that $c_\tau \to 0$ as $\tau \to 0^+$.

Consider now the linear operator $\widetilde{\Psi}_\tau : X \to L^2(0,\tau;Y)$ defined by $\widetilde{\Psi}_\tau x = CT(\cdot)x$. Then admissibility (i.e., $M(\tau)<\infty$) means that $\widetilde{\Psi}_\tau$ is a bounded operator. If in addition $m(\tau)>0$, then $\widetilde{\Psi}_\tau$ is injective and has closed range. Therefore, we may consider the operator $\Psi_\tau : X \to \mathcal{R}(\widetilde{\Psi}_\tau)$, $\Psi_\tau = \widetilde{\Psi}_\tau$. We have

$$\frac{1}{m(\tau)} = \sup_{x\in\mathcal{D}(A),\|x\|=1}\frac{\|x\|}{\|\Psi_\tau x\|} = \sup_{x\in\mathcal{D}(A),\|x\|\neq 0}\frac{\|x\|}{\|\Psi_\tau x\|} = \left\|\Psi_\tau^{-1}\right\|. \tag{2.2}$$

We introduce the following definition.

Definition 2.2. We say that the system (A, C) has the backward–forward conditioning property or shortly that (A, C) is a BFC-system if there exists some $0 < \eta < \tau$ such that C is admissible and exactly observable in time τ and if

$$\left\|\Psi_\tau^{-1}\right\| \|\Psi_\eta\| < 1. \tag{BFC}$$

The condition (BFC) is clearly a conditioning property for the output operator with different times η and τ which correspond to a backward and forward evolution of the system. It also follows from (2.2) that (BFC) is equivalent to

$$M(\eta) < m(\tau) \quad \text{for some } \eta < \tau. \tag{2.3}$$

Therefore, if C is exactly observable in some time τ and of zero-class, then (2.3) holds trivially by letting η sufficiently small. Hence, the system is BFC. If C is admissible at any $\tau > 0$ and if $m(t) \to +\infty$ for $t \to +\infty$, then (2.3) holds and again the system is BFC.

Zero-class admissible operators are introduced and studied in [30]. See also [10] from which we borrow a concrete example leading to an BFC-system in which C is not zero-class.

Example 2.3. The following example is taken from Jacob, Partington and Pott [10, Example 3.9]. We shall use Ingham inequalities to prove that our system is BFC. Similar ideas could be used in a more general class of examples. Consider an undamped wave equation on [0, 1] with Dirichlet boundary conditions and Neumann type observation of the form

$$\begin{cases} \frac{\partial^2}{\partial t^2} z(x,t) = \frac{\partial^2}{\partial x^2} z(x,t) & \text{for } x \in (0,1), t \geqslant 0, \\ z(0,t) = z(1,t) = 0 & \text{for } t \geqslant 0, \\ z(x,0) = z_0(x) \quad \text{and} \quad \frac{\partial}{\partial t} z(x,0) = z_1(x) & \text{for } x \in (0,1), \\ y(t) = \frac{\partial}{\partial x} z(0,t). \end{cases}$$

We rewrite the system as a first order Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} U = -AU(t), & t \geqslant 0, \\ U(0) = (z_0, z_1), \\ CU(x,t) = \frac{\partial}{\partial x} f(0) \end{cases}$$

where $A = \begin{pmatrix} 0 & -I \\ -\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$ and $U = (f, g)$. The latter Cauchy problem is considered on the Hilbert space $H = H_0^1(0,1) \times L^2(0,1)$ endowed with the norm $\|(f,g)\| = \sqrt{\int_0^1 |f'|^2\,dx + \int_0^1 |g|^2\,dx}$. Note that by the Poincaré inequality, $\sqrt{\int_0^1 |f'|^2\,dx}$ defines a norm on $H_0^1(0,1)$ which is equivalent to the usual one.

It is a standard fact that $-A$ generates a strongly continuous semigroup $T(t)$ on H. It is easy to see that A has compact resolvent and the eigenvalues are $\lambda_n = -in\pi$, $n \in \mathbb{Z} \setminus \{0\}$ with normalised eigenfunctions $U_n(x) = (\frac{\sin(n\pi x)}{in\pi}, \sin(n\pi x))$ which form an orthonormal basis of H. Fix $(f, g) \in H$ and denote by $\alpha_n = \langle (f,g), U_n \rangle_H$ (the scalar product in H). Then

$$\left\|(f,g)\right\|^2 = \sum_{n \in \mathbb{Z},\, n \neq 0} |\alpha_n|^2$$

and

$$CT(t)(f,g) = \sum_n \alpha_n e^{in\pi t} CU_n = -i \sum_n \alpha_n e^{in\pi t}.$$

Using the well-known Ingham inequalities (see e.g. [31, p. 162] or [13, Theorem 4.3]) we obtain the following estimates for all $\tau > 2$,

$$m(\tau)^2 \sum_n |\alpha_n|^2 \leqslant \int_0^\tau \left| \sum_n \alpha_n e^{in\pi t} \right|^2 dt \leqslant M(\tau)^2 \sum_n |\alpha_n|^2$$

with

$$m(\tau)^2 \geqslant \frac{2\tau}{\pi}\left(1 - \frac{4}{\tau^2}\right), \qquad M(\tau)^2 \leqslant \frac{8\tau}{\pi}\left(1 + \frac{4}{\tau^2}\right).$$

This shows that C is admissible at any time $\tau > 0$ and exactly observable in time $\tau > 2$ with constant $m(\tau) \to +\infty$ as $\tau \to +\infty$. This shows (2.3) and hence the system (A, C) is backward–forward conditioning.

In order to see that C is not zero-class, we consider small $\tau > 0$ and $f \in H_0^1(0,1)$ with Fourier coefficients α_n and note that

$$\int_0^\tau \left| \sum_n \alpha_n e^{in\pi t} \right|^2 dt = \|\chi_{[0,\tau]} f\|_2^2,$$

where $\chi_{[0,\tau]}$ denote the indicator function of $[0,\tau]$. From this equality it is clear that

$$1 = \sup_{\|f\|_2 = 1} \|\chi_{[0,\tau]} f\|_2^2 = \sup_{\|f\|_2=1} \int_0^\tau \left| \sum_n \alpha_n e^{in\pi t} \right|^2 dt = M(\tau)^2.$$

Therefore, the right-hand side does not converge to 0 as $\tau \to 0$.

Remark 2.4. The fact that the above example is a (BFC) system can also be seen by an abstract argument as follows: let C be admissible in some arbitrary time $\tau > 0$ and exactly observable in some time $\eta > 0$ for a *group* $U(t)_{t\in\mathbb{R}}$. Observe that $\|x\| = \|U(-t)U(t)x\| \leqslant \|U(-t)\| \|U(t)x\|$ whence $\|U(t)x\| \geqslant \|U(-t)\|^{-1} \|x\|$. From

$$\int_0^{n\eta} \|CU(t)x\|^2 dt = \sum_{j=0}^{n-1} \int_0^\eta \|CU(t)U(j\eta)x\|^2 dt \geqslant m(\eta)^2 \left(\sum_{j=0}^{n-1} \|U(-j\eta)\|^{-2} \right) \|x\|^2,$$

we then infer $m(n\eta) \to +\infty$ for $n \to +\infty$ whenever the sum in the last expression diverges. This is in particular the case for bounded groups $U(t)_{t\in\mathbb{R}}$. By the admissibility of the system (A, C) one then obtains (BFC) by letting n sufficiently large.

Let us finally remark that (BFC) is not automatic for exactly observable and admissible systems.

Example 2.5. We consider the left shift semigroup $T(t)$ on $L^2([0,1])$ with point observation operator $Cf = f(0)$. Then

$$\int_0^\tau \left|CT(t)f\right|^2 dt = \|\chi_{[0,\tau)} f\|_2^2.$$

Hence C is admissible and $M(\tau) = 1$ for all $\tau > 0$. In addition, $m(\tau) = 0$ for $\tau < 1$ and $m(\tau) = 1$ for all $\tau \geqslant 1$. Therefore C is admissible and exactly observable at time $\tau = 1$ without being (BFC).

3. Spectral properties of BFC-systems

We consider the same notation X, Y, A, $(T(t))_{t\geqslant 0}$ and $C : D(A) \to Y$ as in the previous section. Our aim here is to study spectral properties of BFC-systems. We will extend some results which have been proved in [30] in the context of zero-class operators. We note also that related ideas and results were obtained previously by Nikolski [21] in the particular case of bounded observation operators C on X.

Let us introduce the classical function $\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\varepsilon(t) := \inf_{\|x\|=1} \left\|T(t)x\right\|.$$

It is clear that $\varepsilon(t)$ is strictly positive for all $t > 0$ if this holds for a single $t_0 > 0$. Indeed, from

$$\left\|T(s)\right\| \left\|T(t)x\right\| \geqslant \left\|T(t+s)x\right\| \geqslant \varepsilon(t)\left\|T(s)x\right\| \geqslant \varepsilon(t)\varepsilon(s)\|x\|$$

one infers that

$$\varepsilon(t)\left\|T(s)\right\| \geqslant \varepsilon(t+s) \geqslant \varepsilon(t)\varepsilon(s),$$

for all $t, s \geqslant 0$. For this reason we distinguish the cases that $\varepsilon(t)$ is strictly positive for all $t > 0$ of that it vanishes for all $t > 0$ and we note this by $\varepsilon(t) > 0$ or $\varepsilon(t) = 0$ respectively.

The following lemma is essentially contained in [21] and [30].

Lemma 3.1. *If (A, C) is an admissible and exactly observable BFC-system, then $\varepsilon(t) > 0$.*

Proof. By the definition of BFC-system, there exist $0 < \eta < \tau$ such that

$$\delta := m(\tau)^2 - M(\eta)^2 > 0.$$

By the semigroup property,

$$m(\tau)^2\|x\|^2 \leqslant \int_0^\tau \|CT(t)x\|_Y^2\,dt = \int_0^\eta \|CT(t)x\|_Y^2\,dt + \int_0^{\tau-\eta} \|CT(t)T(\eta)x\|_Y^2\,dt$$
$$\leqslant M(\eta)^2\|x\|^2 + M(\tau-\eta)^2\|T(\eta)x\|^2$$

which immediately yields

$$\|T(\eta)x\|^2 \geqslant M(\tau-\eta)^{-2}\big(m(\tau)^2 - M(\eta)^2\big)\|x\|^2 \geqslant M(\tau-\eta)^{-2}\delta\|x\|^2.$$

Therefore, $\varepsilon(\eta) > 0$, and hence $\varepsilon(t) > 0$ for all $t > 0$. □

Lemma 3.2. *Suppose that (A, C) is an admissible and exactly observable BFC-system. Then $T(t)^*$ is injective for one (and thus all) $t > 0$ if and only if $T(t)$ extends to a group on X.*

Proof. We know by Lemma 3.1 that $\varepsilon(t) > 0$. This implies that $T(t)$ is injective and has closed image for all $t \geqslant 0$. Thus, $T(t)$ is bijective if and only if $T(t)^*$ is injective. The latter is clearly independent of $t > 0$ by the semigroup law. Indeed, if $T(t_0)^*$ is injective for some $t_0 > 0$, so are all $T(s)^*$ for $s < t_0$ since $T(t_0)^* = T(t_0 - s)^*T(s)^*$. If $s > t_0$ then we find $n \in \mathbb{N}$, $\delta \in [0, t_0[$ such that $T(s)^* = (T(t_0)^*)^n T(\delta)^*$, and the injectivity of $T(s)^*$ follows from that of $T(t_0)^*$ and $T(\delta)^*$. We saw that $T(t)^*$ is injective for one (and thus all) $t > 0$ if and only $T(t)$ is bijective which in turn by

$$S(t) := \begin{cases} T(t) & \text{if } t \geqslant 0, \\ T(t)^{-1} & \text{if } t < 0 \end{cases}$$

is equivalent to a group extension of $T(t)$ on X. □

For a closed operator S on X recall the notions of point spectrum

$$\sigma_P(S) = \big\{\lambda \in \mathbb{C}\colon \ker(\lambda I - S) \neq \{0\}\big\},$$

the approximate point spectrum

$$\sigma_A(S) = \Big\{\lambda \in \mathbb{C}\colon \inf_{x\in\mathcal{D}(S), \|x\|=1} \|\lambda x - Sx\| = 0\Big\}$$

and the residual spectrum

$$\sigma_R(S) = \big\{\lambda \in \mathbb{C}\colon \text{range } (\lambda - S) \text{ is not dense in } X\big\}.$$

It is easy to see that $\sigma_R(S) \cup \sigma_A(S) = \sigma(S)$. Of course, $\sigma_P(S) \subseteq \sigma_A(S)$.

Proposition 3.3. *Let (A, C) be an admissible and exactly observable BFC-system. Then there exist no approximate point spectrum of A with arbitrary large real parts.*

In particular, if C is an admissible zero-class operator and A has a sequence of approximate point spectrum with arbitrary large real parts then C is not exactly observable.

Proof. Recall that $\exp(-t\sigma_A(A)) \subseteq \sigma_A(T(t))$ (see [6, p. 276], note that $-A$ is the generator). If we find a sequence $\lambda_n \in \sigma_A(A)$ with $\mathrm{Re}(\lambda_n) \to +\infty$, then $e^{-t\lambda_n} \in \sigma_A(T(t))$. Hence, $\inf_{\|x\|=1} \|T(t)x - e^{-t\lambda_n}x\| = 0$. By

$$\varepsilon(t)\|x\| \leqslant \|T(t)x\| \leqslant \|T(t)x - e^{-t\lambda_n}x\| + e^{-t\,\mathrm{Re}(\lambda_n)}\|x\|$$

we get $\varepsilon(t) = 0$ which is incompatible with exact observability by Lemma 3.1. □

Since $-A$ is the generator of a strongly continuous semigroup, $\sigma(A)$ is contained in a right-half plane. On the other hand, it is well known that the boundary of the spectrum $\partial\sigma(A)$ is contained in $\sigma_A(A)$. We obtain from the previous proposition that $\mathrm{Re}(\partial\sigma(A))$ is bounded, i.e., $\partial\sigma(A)$ is contained in a vertical strip. Thus

Corollary 3.4. *Let (A, C) be an admissible BFC-system. Then*

$$\mathrm{Re}\big(\partial\sigma(A)\big) := \big\{\mathrm{Re}(\lambda)\colon \lambda \in \partial\sigma(A)\big\}$$

is bounded.

The following lemma is known.

Lemma 3.5. *Let $S \in \mathcal{B}(X)$ satisfy $\|Sx\| \geqslant \gamma\|x\|$ for some $\gamma > 0$ and all $x \in X$ and assume $0 \in \sigma(S)$. Then there exists $\delta > 0$ such that $B(0,\delta) \subseteq \sigma_R(S)$.*

The main result in [30] which states that if C is zero-class admissible and $\sigma_R(A)$ is empty, then $T(t)$ extends to a group. The next propositions extend this result.

Proposition 3.6. *Let (A, C) be an admissible BFC-system. If $\mathrm{Re}(\sigma_R(A)) := \{\mathrm{Re}\,\lambda\colon \lambda \in \sigma_R(A)\}$ is bounded, then $(T(t))_{t\geqslant 0}$ extends to a group on X.*

Proof. We know by Lemma 3.1 that $\varepsilon(t) > 0$. If $T(t)$ was not boundedly invertible for some $t > 0$, then $0 \in \sigma(T(t))$. By Lemma 3.5, there exists $\delta_t > 0$ such that $B(0,\delta_t) \subseteq \sigma_R(T(t))$. Since $\sigma_R(T(t))\backslash\{0\} = \exp(-t\sigma_R(A))$ (see [6, p. 276]) we obtain

$$B(0,\delta_t)\backslash\{0\} \subseteq \exp\big(-t\sigma_R(A)\big).$$

Therefore, there exists a real sequence $(\lambda_n) \in \sigma_R(A)$ such that $\mathrm{Re}\,\lambda_n \to +\infty$. This contradicts the assumption. □

Proposition 3.7. *Assume that (A, C) is an admissible BFC-system. If $\sigma(A)$ does not contain a half-plane then $(T(t))_{t\geqslant 0}$ extends to a group on X.*

Proof. By Corollary 3.4 we see that $\sigma(A)$ is either contained in vertical strip or contains a half-plane. Now we apply Proposition 3.6 to conclude. □

Considering the right shift semigroup $T(t)$ on $L^2(\mathbb{R}_+)$ with the identity observation $C = \mathrm{Id}$ provides an example of a BFC-system (even a zero-class admissible one, see [30, Remark 3.1]) for which no group extension is possible. In this example, it is not difficult to check that the spectrum of A satisfies $\sigma(A) = \sigma_R(A) = \mathbb{C}^+$ (the right half-plane). This shows that the spectral condition in Proposition 3.7 cannot be omitted.

Assume that (A, C) is an admissible BFC-system. If $T(t)$ is analytic, differentiable or merely eventually continuous then by [6, p. 113] $\sigma(A)$ does not contain a half-plane. Thus, we conclude by Proposition 3.7 that $(T(t))_{t\geqslant 0}$ extends to group on X.

Proposition 3.8. *Assume that (A, C) is an admissible BFC-system. If $T(t)$ is compact for some $t > 0$, then X has finite dimension.*

Proof. If $T(t)$ is compact for some $t > 0$ then $\sigma(A) = \sigma_P(A)$ is discrete. It follows from Proposition 3.3 that $\sigma(A)$ is bounded. We conclude by Proposition 3.6 that $(T(t))_{t\geqslant 0}$ extends to a group on X. Thus, $I = T(t)T(-t)$ is compact on X and therefore X has finite dimension. □

4. Sufficient conditions for exact or final state observability

Our aim in this section is to derive conditions on C and A which imply exact or final state observability. Our condition reads as follows

$$\left\| CA^{-(1-\beta)}x \right\|_Y \geqslant \delta \|x\| \tag{4.1}$$

for all $x \in \mathcal{D}(A)$. Here $\beta \in (0, 1)$ and $\delta > 0$ are constants. We shall assume that A is injective and hence (4.1) is understood in the sense

$$\|Cx\|_Y \geqslant \delta \left\| A^{1-\beta}x \right\|. \tag{4.2}$$

This makes sense for $x \in \mathcal{D}(A)$ since $\mathcal{D}(A) \subset \mathcal{D}(A^{1-\beta})$.

In the sequel we need some basic properties of the H^∞ functional calculus for sectorial operators. This functional calculus goes back to the work of McIntosh [18]. More recent publications of the meanwhile rich theory can be found in [7] or [14] and references given therein. We briefly sketch the needed results and definitions.

Definition 4.1. We denote by S_ω the open sector $\{z \in \mathbb{C}^*\colon\ |\arg(z)| < \omega\}$ and by $\overline{S_\omega}$ the closure of S_ω in $\mathbb{C}$.

We call a closed operator A on X *sectorial of angle ω* if A is densely defined having its spectrum in $\overline{S_\omega}$ such that $\lambda R(\lambda, A) := \lambda(\lambda - A)^{-1}$ of A is uniformly bounded on the complement of each strictly larger sectors S_θ, $\theta > \omega$.

Notice that if $-A$ generates a bounded semigroup $T(t)_{t\geqslant 0}$, then A is sectorial of angle $\frac{\pi}{2}$ by the Hille–Yosida theorem. Moreover, the semigroup is bounded analytic if and only if A is sectorial of angle $< \frac{\pi}{2}$.

Let $H^\infty(S_\theta)$ be the set of bounded holomorphic functions on S_θ and $H^\infty(\overline{S_\omega})$ the set of holomorphic functions on S_ω that are continuous and bounded on $\overline{S_\omega}$. We further consider the ideal $H_0^\infty(S_\theta)$ (respectively $H_0^\infty(\overline{S_\omega})$) of all functions $f \in H^\infty(S_\theta)$ (respectively $H_0^\infty(\overline{S_\omega})$) that allow an estimate $|f(z)| \leqslant M \max(|z|^\varepsilon, |z|^{-\varepsilon})$. Given a sectorial operator A of angle ω and $f \in H_0^\infty(S_\theta)$ for some $\theta > \omega$ we define the Cauchy integral

$$f(A) := \frac{1}{2\pi i} \int_{\Gamma} f(\lambda) R(\lambda, A)\, d\lambda$$

where and $\Gamma = \partial S_\theta$ denotes the oriented path with strictly decreasing imaginary parts. This integral converges absolutely in norm and defines therefore a bounded operator $f(A)$.

We say that A admits a bounded $H^\infty(S_\theta)$ functional calculus if there exists a constant M such that

$$\big\| f(A) \big\| \leqslant M \| f \|_\infty \tag{4.3}$$

for all $f \in H^\infty(S_\theta)$.

It is known that by an approximation argument, if (4.3) holds for all $f \in H_0^\infty(S_\theta)$ then it holds for $f \in H^\infty(S_\theta)$.

In some situations one can define a holomorphic functional calculus $f(A)$ for $f \in H_0^\infty(\overline{S_\omega})$ and A is a sectorial operator of angle ω. For example this is the case for accretive operators on Hilbert spaces. See [4,7,2] and references therein.

We say that A admits upper square function estimate on S_θ if there is a constant $M > 0$ such that

$$\forall x \in X\colon \quad \int_0^\infty \big\| \varphi(tA)x \big\|^2 \frac{dt}{t} \leqslant M^2 \|x\|^2 \tag{4.4}$$

for all $\varphi \in H_0^\infty(S_\theta)$. In the same way, a lower square function estimate on S_θ corresponds to the estimate

$$\forall x \in X\colon \quad \|x\|^2 \leqslant K^2 \int_0^\infty \big\| \varphi(tA)x \big\|^2 \frac{dt}{t}. \tag{4.5}$$

If $X = H$ is a Hilbert space, then upper square function estimates for A and for A^* hold for all functions in $H_0^\infty(S_\theta)$ and all $\theta > \omega$ provided A has a bounded $H_0^\infty(S_\theta)$ functional calculus. Moreover, by a duality argument, lower square estimates for A follow from upper estimates of the adjoint operator A^*. We refer at this point to [18,3] for more information. We will go into some details in the last section of this paper.

Before stating our first result of this section we discuss the following estimate for A

$$\|x\|^2 \leqslant K^2 \int_0^\infty \big\| (tA)^{-\beta} \big(T\big(2t^{2\beta}\big) - T\big(t^{2\beta}\big)\big) x \big\|^2 \frac{dt}{t} \tag{SQ_β}$$

that we will need to formulate the theorem. Here, K is a positive constant and $\beta \in (0, 1)$. In case $\beta = \frac{1}{2}$, letting $\varphi(z) := z^{-\frac{1}{2}}(e^{-2z} - e^{-z})$, this corresponds to a lower square function estimate (4.5) for $\varphi \in H_0^\infty(\overline{S_{\frac{\pi}{2}}})$. As mentioned above, (4.5) follows from H^∞-functional calculus when X is a Hilbert space. We will discuss again this in the next section and will consider the limit case where A is sectorial of angle $\frac{\pi}{2}$, the above function φ is not bounded on $S_{\frac{\pi}{2}+\varepsilon}$ for any $\varepsilon > 0$.

Assume that $-A$ is the generator of bounded strongly continuous semigroup on X and is injective. If we let $\varphi(z) := z^{-\beta}(e^{-2z} - e^{-z})$, then (SQ_β) can be seen as

$$\begin{aligned}\|x\|^2 &\leqslant K^2 \int_0^\infty \bigl\|(tA)^{-\beta}\bigl(T\bigl(2t^{2\beta}\bigr) - T\bigl(t^{2\beta}\bigr)\bigr)x\bigr\|^2 \frac{dt}{t} \\ &= K^2 \int_0^\infty \bigl\|t^{2\beta^2-\beta}\varphi\bigl(t^{2\beta}A\bigr)x\bigr\|^2 \frac{dt}{t} \\ &= \frac{K^2}{2\beta} \int_0^\infty \bigl\|\varphi(sA)x\bigr\|^2 s^{-(1-2\beta)} \frac{ds}{s},\end{aligned}$$

i.e., as 'weighted' lower square function estimate for $\varphi \in H_0^\infty(\overline{S_{\frac{\pi}{2}}})$.

By [7, Theorem 6.4.6], the completion $X_{\frac{1}{2}-\beta,2}$ of X with respect to the semi-norm

$$[x]_\beta = \left(\int_0^\infty \bigl\|\varphi(sA)x\bigr\|^2 s^{-(1-2\beta)} \frac{ds}{s}\right)^{\frac{1}{2}}$$

is independent of the choice of φ and coincides (with equivalent norms) with the real interpolation space:

$$\bigl(\dot{X}_{-1}(A), \dot{X}_1(A)\bigr)_{\frac{3}{4}-\frac{\beta}{2},2} = X_{\frac{1}{2}-\beta,2}. \tag{4.6}$$

Here $\dot{X}_{-1}(A)$ is the completion of $\mathcal{R}(A)$ with respect to $\|A^{-1}x\|$ and $\dot{X}_1(A)$ is the completion of $\mathcal{D}(A)$ with respect to $\|Ax\|$. From (4.6), it follows that (SQ_β) is equivalent to the continuous embedding

$$\bigl(\dot{X}_{-1}(A), \dot{X}_1(A)\bigr)_{\frac{3}{4}-\frac{\beta}{2},2} \hookrightarrow X. \tag{4.7}$$

For the rest of this discussion, we assume for simplicity that A is invertible. In this case, $\dot{X}_1(A) = \mathcal{D}(A)$ and $\dot{X}_{-1}(A) = X_{-1}$.

It is a known fact that the semigroup $(T(t))$ extends to a strongly continuous semigroup $(T_{-1}(t))$ on the extrapolation space X_{-1}, whose (negative) generator A_{-1} is an extension of A (see [6, Chapter II, Section 5]). In addition A is the part of A_{-1} on X and hence $\mathcal{D}(A_{-1}^2) = \mathcal{D}(A)$ with equivalent norms. Indeed,

$$\begin{aligned} x \in \mathcal{D}\bigl(A_{-1}^2\bigr) &\Leftrightarrow x \in \mathcal{D}(A_{-1}) = X, \quad A_{-1}x \in \mathcal{D}(A_{-1}) \\ &\Leftrightarrow x \in X, \quad A_{-1}x \in \mathcal{D}(A_{-1}) \\ &\Leftrightarrow x \in \mathcal{D}(A). \end{aligned}$$

The fact that $A_{-1} : X \to X_{-1}$ is an isometry implies that $\|Ax\|_X = \|A_{-1}^2 x\|_{X_{-1}}$.

Assume now that $\beta < \frac{1}{2}$. We have

$$\begin{aligned}\big(X_{-1}, \mathcal{D}(A)\big)_{\frac{3}{4}-\frac{\beta}{2},2} &= \big(X_{-1}, \mathcal{D}\big(A_{-1}^2\big)\big)_{\frac{3}{4}-\frac{\beta}{2},2}\\ &= \big(\mathcal{D}(A_{-1}), \mathcal{D}\big(A_{-1}^2\big)\big)_{\frac{1}{2}-\beta,2}\\ &= \big(X, \mathcal{D}(A)\big)_{\frac{1}{2}-\beta,2} \hookrightarrow X.\end{aligned}$$

Note that the second equality follows from [28, p. 105]. Hence for $\beta < \frac{1}{2}$

$$X_{\frac{1}{2}-\beta,2} = \big(X_{-1}, \mathcal{D}(A)\big)_{\frac{3}{4}-\frac{\beta}{2},2} \hookrightarrow X, \tag{4.8}$$

which means that (SQ_β) always holds for $\beta < \frac{1}{2}$.

Let us finally mention that for $\beta > \frac{1}{2}$, (SQ_β) never holds for non-negative self-adjoint operators with compact resolvent in infinite dimension separable Hilbert spaces. Indeed, consider such an operator A. The spectrum is discrete $\sigma(A) = \{\lambda_n\}$ with $\lambda_n \to +\infty$. Applying (SQ_β) to a normalised eigenvector $x = x_n$ (associated with λ_n) yields

$$\begin{aligned}1 &\leqslant K^2 \int_0^\infty \big\|(tA)^{-\beta} x_n\big\|^2 \big(e^{-2t^{2\beta}} - e^{-t^{2\beta}}\big)^2 \frac{dt}{t}\\ &= \frac{K^2}{\lambda_n^{2\beta}} \int_0^\infty t^{-2\beta} \big(e^{-2t^{2\beta}\lambda_n} - e^{-t^{2\beta}\lambda_n}\big)^2 \frac{dt}{t}\\ &= \frac{K^2\lambda_n}{2\beta\lambda_n^{2\beta}} \int_0^\infty \big(e^{-2s} - e^{-s}\big)^2 \frac{ds}{s^2}\\ &= C\lambda_n^{1-2\beta}.\end{aligned}$$

For $\beta > \frac{1}{2}$, the last term goes to 0 as n tends to $+\infty$. This shows that (SQ_β) cannot hold.

Now we come to our main results of this section. We start with exact observability.

Theorem 4.2. *Let $-A$ be the generator of a bounded semigroup on a Banach space X and assume that A is injective. Let $C : \mathcal{D}(A) \to Y$ be bounded and suppose that there exists $\beta \in (0, 1)$ such that the lower square function estimate (SQ_β) and* (4.1) *are satisfied. Then C is exactly observable for A in infinite time, i.e., there exists a constant $m > 0$ such that*

$$m^2\|x\|^2 \leqslant \int_0^\infty \big\|CT(t)x\big\|_Y^2\, dt \tag{4.9}$$

for all $x \in \mathcal{D}(A)$.

Proof. Fix $x \in \mathcal{D}(A)$ and apply the lower square function estimate (SQ_β) to obtain

$$\begin{aligned}
\|x\|^2 &\leqslant K^2 \int_0^\infty \big\|(tA)^{-\beta}\big(T\big(2t^{2\beta}\big) - T\big(t^{2\beta}\big)\big)x\big\|^2 \frac{dt}{t} \\
&= K^2 \int_0^\infty \left\| t^{-\beta} A^{(1-\beta)} \int_{t^{2\beta}}^{2t^{2\beta}} T(s)x\,ds \right\|^2 \frac{dt}{t} \\
&\leqslant \frac{K^2}{\delta^2} \int_0^\infty \left\| C \int_{t^{2\beta}}^{2t^{2\beta}} T(s)x\,ds \right\|_Y^2 \frac{dt}{t^{1+2\beta}}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x\|^2 &\leqslant \frac{K^2}{\delta^2} \int_0^\infty \left(\int_{t^{2\beta}}^{2t^{2\beta}} 1 \cdot \big\|CT(s)x\big\|_Y\,ds \right)^2 \frac{dt}{t^{1+2\beta}} \\
&\leqslant \frac{K^2}{\delta^2} \int_0^\infty \int_{t^{2\beta}}^{2t^{2\beta}} \big\|CT(s)x\big\|_Y^2\,ds\,\frac{dt}{t} \\
&= \frac{K^2}{\delta^2} \int_1^2 \int_0^\infty \big\|CT\big(t^{2\beta}u\big)x\big\|_Y^2 t^{2\beta-1}\,dt\,du \\
&= \log(2) \frac{K^2}{2\beta\delta^2} \int_0^\infty \big\|CT(r)x\big\|_Y^2\,dr.
\end{aligned}$$

This shows (4.9) with $m^{-1} = \sqrt{\log(2)/2\beta}\,\frac{K}{\delta}$. $\square$

In [17], Le Merdy shows an admissibility result (the Weiss conjecture for analytic semigroups) using upper square function estimates. Here, we use lower square function estimates to obtain an observability result. In this sense our result represents a kind of counterpart to [17]. However, Le Merdy does need analyticity of the semigroup whereas we don't.

Remark 4.3. It is straightforward to generalise the previous result to L^p-observability using L^p-square function estimates and Hölder's inequality. The case $p = 1$ and $\beta = \frac{1}{2}$ is of interest since L^1-square function estimates hold trivially for *any* semigroup using (5.2) below. By dualising to the control situation this corresponds to an L^∞ control.

In the next corollary we obtain a criterion for finite time exact observability.

Corollary 4.4. *Let $-A$ be the generator of a bounded semigroup on the Banach space X and assume that there exists a constant $\omega > 0$ such that $A + \omega$ satisfies (SQ_β). Assume that $C : \mathcal{D}(A) \to Y$ is bounded and that*

$$\left\| C(\omega + A)^{-(1-\beta)} x \right\|_Y \geqslant \delta \|x\|$$

for $x \in \mathcal{D}(A)$ and that C is infinite-time admissible for the semigroup $e^{-\omega t}T(t)$. Then C is exactly observable in finite time.

Proof. We apply the previous theorem to $\omega + A$ and obtain

$$\|x\|^2 \leqslant M \int_0^\infty \left\| Ce^{-\omega t} T(t)x \right\|_Y^2 dt$$

for all $x \in \mathcal{D}(A)$. We split the right-hand side into two parts and write

$$\begin{aligned}
\int_0^\infty \left\| Ce^{-\omega t} T(t)x \right\|_Y^2 dt &= \int_0^\tau \left\| Ce^{-\omega t} T(t)x \right\|_Y^2 dt + \int_\tau^\infty \left\| Ce^{-\omega t} T(t)x \right\|_Y^2 dt \\
&\leqslant \int_0^\tau \left\| CT(t)x \right\|_Y^2 dt + \int_0^\infty \left\| Ce^{-\omega(\tau+t)} T(t+\tau)x \right\|_Y^2 dt \\
&= \int_0^\tau \left\| CT(t)x \right\|_Y^2 dt + e^{-\omega\tau} \int_0^\infty \left\| Ce^{-\omega t} T(t)T(\tau)x \right\|_Y^2 dt.
\end{aligned}$$

Since C is infinite-time admissible for $e^{-\omega t}T(t)$ and the semigroup $T(t)$ is bounded we have for some constants M', M''

$$\int_0^\infty \left\| Ce^{-\omega t} T(t)T(\tau)x \right\|_Y^2 dt \leqslant M' \left\| T(\tau)x \right\|^2 \leqslant M'' \|x\|^2.$$

Therefore,

$$\|x\|^2 \leqslant M \int_0^\tau \left\| CT(t)x \right\|_Y^2 dt + MM'' e^{-\omega\tau} \|x\|^2.$$

If we choose τ large enough such that $MM''e^{-\omega\tau} < 1$ we obtain the desired inequality. □

As explained at the beginning of this section, (SQ_β) holds for all $\beta < \frac{1}{2}$ and all generators of bounded semigroup (see (4.8)). Therefore, applying Theorem 4.2 with $\beta = \frac{1}{2} - \varepsilon$, we obtain the following corollary.

Corollary 4.5. *Let $-A$ be the generator of a bounded semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X and assume that A is injective. Then, if $\|CA^{-\frac{1}{2}+\varepsilon}x\| \geqslant \delta\|x\|$ for some $\varepsilon, \delta > 0$ and all $x \in \mathcal{D}(A)$, C is infinite-time exactly observable for A.*

In Theorem 4.2 we assume that A and C satisfy (4.1) and obtain exact observability. Now if instead of (4.1) we assume that for some time $\tau_0 > 0$

$$\left\|CA^{-(1-\beta)}x\right\|_Y \geqslant \delta\left\|T(\tau_0)x\right\| \tag{4.10}$$

then we obtain final state observability. For simplicity, we formulate the following result for the case $\beta = \frac{1}{2}$.

Theorem 4.6. *Suppose that $-A$ generates an exponentially stable semigroup on the Banach space X which satisfies (SQ_β) with $\beta = \frac{1}{2}$ and the upper square function estimate* (4.4) *for all $\varphi \in H_0^\infty(\overline{S_{\frac{\pi}{2}}})$. Let $C : \mathcal{D}(A) \to Y$ be bounded and suppose that there exists $\tau_0 > 0$ such that*

$$\left\|CA^{-\frac{1}{2}}x\right\|_Y \geqslant \delta\left\|T(\tau_0)x\right\| \tag{4.11}$$

for all $x \in \mathcal{D}(A)$. Then (A, C) is final state observable, i.e., there exists a constant $m > 0$ and $\tau > 0$ such that

$$m^2\left\|T(\tau)x\right\|^2 \leqslant \int_0^\tau \left\|CT(t)x\right\|_Y^2\,dt \tag{4.12}$$

for all $x \in \mathcal{D}(A)$.

Proof. Let $\tau > \tau_0$ and fix $x \in \mathcal{D}(A)$. By the lower square function estimate we have

$$\left\|T(\tau)x\right\|^2 \leqslant K^2 \int_0^\infty \left\|(tA)^{-\beta}\big(T(2t) - T(t)\big)T(\tau)x\right\|^2 \frac{dt}{t}. \tag{4.13}$$

We split the integral into two parts $\int_0^\tau + \int_\tau^\infty$. Using the fact that $\|T(\tau)x\| \leqslant M\|T(\tau_0)x\|$ we can estimate the first term as follows

$$\begin{aligned}
\int_0^\tau \left\|(tA)^{-\frac{1}{2}}\big(T(2t) - T(t)\big)T(\tau)x\right\|^2 \frac{dt}{t} &= \int_0^\tau \left\|T(\tau)A^{\frac{1}{2}}\int_t^{2t} T(s)x\,ds\right\|^2 \frac{dt}{t^2} \\
&\leqslant M^2 \int_0^\tau \left\|T(\tau_0)A^{\frac{1}{2}}\int_t^{2t} T(s)x\,ds\right\|^2 \frac{dt}{t^2}
\end{aligned}$$

$$\leqslant M^2\delta^{-2}\int_0^{\tau}\left\|\int_t^{2t} CT(s)x\,ds\right\|_Y^2\frac{dt}{t^2}.$$

The same calculations as in the proof of Theorem 4.2 show that the latter term is bounded by

$$K_0\int_0^{2\tau}\left\|CT(r)x\right\|_Y^2\,dr,$$

where K_0 is a positive constant. Hence

$$\int_0^{\tau}\left\|(tA)^{-\frac{1}{2}}\big(T(2t)-T(t)\big)T(\tau)x\right\|^2\frac{dt}{t}\leqslant K_0\int_0^{2\tau}\left\|CT(r)x\right\|_Y^2\,dr. \tag{4.14}$$

Now we consider the second term. Set $\varphi(z)=z^{\frac{1}{2}}(e^{-z}-1)$. We apply the upper square function estimate and obtain

$$\begin{aligned}
\int_{\tau}^{\infty}\left\|(tA)^{-\frac{1}{2}}\big(T(2t)-T(t)\big)T(\tau)x\right\|^2\frac{dt}{t} &= \int_{\tau}^{\infty}\left\|T(t)(tA)^{-\frac{1}{2}}\big(T(t)-I\big)T(\tau)x\right\|^2\frac{dt}{t}\\
&\leqslant M^2\left\|T(\tau)\right\|^2\int_{\tau}^{\infty}\left\|(tA)^{-\frac{1}{2}}\big(T(t)-I\big)T(\tau)x\right\|^2\frac{dt}{t}\\
&= M^2\left\|T(\tau)\right\|^2\int_{\tau}^{\infty}\left\|\varphi(tA)T(\tau)x\right\|^2\frac{dt}{t}\\
&\leqslant M'\left\|T(\tau)\right\|^2\left\|T(\tau)x\right\|^2.
\end{aligned}$$

From this, (4.13) and (4.14) we obtain that there exists a positive constant M'' (independent of τ) such that

$$\left\|T(\tau)x\right\|^2\leqslant M''\int_0^{2\tau}\left\|CT(r)x\right\|_Y^2\,dr+M''\left\|T(\tau)\right\|^2\left\|T(\tau)x\right\|^2.$$

Since the semigroup is supposed to be exponentially stable there exists $\tau>0$ such that $\|T(\tau)\|<1$ and $M''\|T(\tau)\|^2\leqslant\frac{1}{2}$ and hence

$$\left\|T(2\tau)x\right\|^2\leqslant\left\|T(\tau)x\right\|^2\leqslant 2M''\int_0^{2\tau}\left\|CT(r)x\right\|_Y^2\,dr.$$

This shows that the system is final state observable at time 2τ. $\square$

In the next section we shall see that the upper and lower square function estimates of Theorem 4.6 are always satisfied for contraction semigroups on Hilbert spaces.

5. Exact observability on Hilbert spaces

Proposition 5.1. *Let X and Y be Hilbert spaces. Then, if (A, C) is exactly observable and admissible in infinite time, $T(t)_{t\geqslant 0}$ is similar to a contraction semigroup.*

Proof. Denote by $\langle x, y\rangle_Y$ the scalar product of Y and define for $x, y \in \mathcal{D}(A)$

$$\langle x, y\rangle_X^\sim := \int_0^\infty \langle CT(t)x, CT(t)y\rangle_Y \, dt.$$

This is clearly a bilinear (or sesquilinear) form on $\mathcal{D}(A) \times \mathcal{D}(A)$. Admissibility and exact observability imply that $\|x\|_X$ and $\|x\|_X^\sim$ are equivalent. By density, we extend this to all $x \in X$ and $\|\cdot\|_X^\sim$ is associated with a scalar product on X. With respect to the new norm,

$$\|T(t)x\|_X^\sim = \left(\int_0^\infty \|CT(t+s)x\|^2 \, ds\right)^{\frac{1}{2}} = \left(\int_t^\infty \|CT(s)x\|^2 \, ds\right)^{\frac{1}{2}} \leqslant \|x\|_X^\sim$$

and so $T(t)_{t\geqslant 0}$ is a contraction semigroup with respect to $\|\cdot\|_X^\sim$. □

For contraction semigroups on Hilbert spaces we have the following result on upper and lower square function estimates.

Theorem 5.2. *Suppose that $-A$ is injective and generates a contraction semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space X.*

(a) *For every $\varphi \in H_0^\infty(\overline{S_{\frac{\pi}{2}}})$, the upper square function estimate* (4.4) *holds.*
(b) *The lower square function estimate holds for the function $\varphi(z) = z^{-\frac{1}{2}}(e^{-2z} - e^{-z})$ and all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. In other words, (SQ_β) holds for $\beta = \frac{1}{2}$ and all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$.*

Proof. Firstly by the Lumer–Phillips theorem, A is an accretive operator, i.e., $\operatorname{Re}\langle Ax, x\rangle_X \geqslant 0$ for all $x \in \mathcal{D}(A)$. Secondly notice that A has a bounded $H_0^\infty(\overline{S_{\frac{\pi}{2}}})$ functional calculus. This means that for every H_0^∞ function φ on $S_{\frac{\pi}{2}}$ that is continuous on $\overline{S_{\frac{\pi}{2}}}$, $\varphi(A)$ one has the estimate

$$\|\varphi(A)\|_{\mathcal{L}(X)} \leqslant M \sup_{z\in\overline{S_{\frac{\pi}{2}}}} |\varphi(z)|.$$

This is essentially von Neumann's inequality for contractions on a Hilbert space. Indeed, if A is accretive, $T := (A-1)(A+1)^{-1}$ is a contraction and von Neumann's inequality states $\|p(T)\| \leqslant \|p\|_{H^\infty(\mathbb{D})}$ for every polynomial p. From this, the H^∞-calculus can be derived by approximation arguments. Two different direct proofs for the boundedness of the $H_0^\infty(\overline{S_{\frac{\pi}{2}}})$ calculus are given in

[7, Theorem 7.1.7]. One uses a dilation theorem of the semigroup into a unitary C_0-group due to Szőkefalvi-Nagy. The second exploits accretivity of A from a 'numerical range' viewpoint and can be seen as the most simple case of the Crouzeix–Delyon theorems [4,2].

Having the boundedness of the functional calculus on $\overline{S_{\frac{\pi}{2}}}$ in hands we certainly have upper square function estimates for functions in $H_0^\infty(S_\theta)$ when $\theta > \frac{\pi}{2}$. This is well known and proved by McIntosh [18]. For the particular functions $\psi_\alpha(z) := z^\alpha/(1+z)$ for $\alpha \in (0,1)$ this means that for some positive constants k_α

$$k_\alpha \int_0^\infty \left\| \psi_\alpha(tA)x \right\|^2 \frac{dt}{t} \leqslant \|x\|^2. \tag{5.1}$$

Given now a function $\varphi \in H_0^\infty(\overline{S_{\frac{\pi}{2}}})$, we choose $\varepsilon > 0$ small such that $|\varphi(z)| \leqslant M \max(|z|^{2\varepsilon}, |z|^{-2\varepsilon})$ and write

$$\varphi(z) = \psi_\varepsilon(z) \times z^{-\varepsilon}\varphi(z) + \psi_{1-\varepsilon}(z) \times z^\varepsilon \varphi(z).$$

Notice that $z^{\pm\varepsilon}\varphi(z) \in H_0^\infty(\overline{S_{\frac{\pi}{2}}})$. Therefore, upper square function estimate for A with the function φ follow from (5.1) and the boundedness of the $H_0^\infty(\overline{S_{\frac{\pi}{2}}})$ calculus. This proves assertion (a).

In order to prove the second assertion we first prove that there exists a constant $c > 0$ such that for every $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$

$$cx = \int_0^\infty (tA)^{-1} \big(T(2t) - T(t)\big)^2 x \frac{dt}{t}. \tag{5.2}$$

We have for fixed $\varepsilon > 0$

$$\begin{aligned}
\int_0^\infty (tA)^{-1}\big(T(2t) - T(t)\big)^2 x e^{-\varepsilon t} \frac{dt}{t} &= \int_0^\infty A \int_t^{2t} T(s)\, ds \int_t^{2t} T(r)x\, dr\, e^{-\varepsilon t} \frac{dt}{t^2} \\
&= \int_0^\infty A \int_1^2 T(tu)\, du \int_1^2 T(vt)x\, dv\, e^{-\varepsilon t}\, dt \\
&= \int_1^2 \int_1^2 \int_0^\infty AT(s)xe^{-\varepsilon \frac{s}{u+v}}\, ds \,\frac{du\, dv}{u+v} \\
&= \int_1^2 \int_1^2 A\left(\frac{\varepsilon}{u+v} + A\right)^{-1} x \frac{du\, dv}{u+v}.
\end{aligned}$$

Now for $x \in \mathcal{R}(A)$ we have $\lambda(\lambda + A)^{-1}x = \lambda(\lambda + A)^{-1}Ax_0 \to 0$ as $0 < \lambda \to 0$ and hence $A(\frac{\varepsilon}{u+v} + A)^{-1}x \to x$ as $\varepsilon \to 0$. In addition, the semigroup $T(t)$ is of contractions and hence

$\|A(\frac{\varepsilon}{u+v}+A)^{-1}x\| \leqslant 2\|x\|$. Note also that $(tA)^{-1}(T(2t)-T(t))^2x$ is in $L^2(0,\infty,\frac{dt}{t})$ by assertion (a). Taking $\varepsilon \to 0$ we conclude by the dominated convergence theorem that

$$\int_0^\infty (tA)^{-1}\bigl(T(2t)-T(t)\bigr)^2 x\frac{dt}{t} = \int_1^2\int_1^2 \frac{du\,dv}{u+v}x = cx.$$

Following an idea in [18] we can use (5.2) to prove a lower square function estimate. For a fixed $y \in X$ we write

$$\begin{aligned}
c\bigl|\langle x,y\rangle_X\bigr| &= \left|\int_0^\infty \bigl\langle (tA)^{-1}\bigl(T(2t)-T(t)\bigr)^2x, y\bigr\rangle_X \frac{dt}{t}\right| \\
&= \left|\int_0^\infty \bigl\langle (tA)^{-\frac{1}{2}}\bigl(T(2t)-T(t)\bigr)x, \bigl(tA^*\bigr)^{-\frac{1}{2}}\bigl(T(2t)^*-T(t)^*\bigr)y\bigr\rangle_X \frac{dt}{t}\right| \\
&\leqslant \left(\int_0^\infty \bigl\|\varphi(tA)x\bigr\|^2\frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_0^\infty \bigl\|\varphi\bigl(tA^*\bigr)y\bigr\|^2\frac{dt}{t}\right)^{\frac{1}{2}}.
\end{aligned}$$

Now we use the upper square estimate of assertion (a) to the adjoint A^* and obtain

$$c\bigl|\langle x,y\rangle_X\bigr| \leqslant K\left(\int_0^\infty \bigl\|\varphi(tA)x\bigr\|^2\frac{dt}{t}\right)^{\frac{1}{2}}\|y\|.$$

Since y is arbitrary we obtain the lower square function estimate. □

Now we turn back to Theorems 4.2 and 4.6. As mentioned at the beginning of the previous section, the lower square function estimate (SQ_β) holds for small β for all generators of bounded strongly continuous semigroups. It is then tempting to use the theorem for small β in order to include a large class of semigroups $T(t)$. On the other hand, if we assume that $0 \in \varrho(A)$ and $\|CA^{-\alpha-\varepsilon}x\| \geqslant \delta\|x\|$, one also has

$$\bigl\|CA^{-\alpha}x\bigr\| = \bigl\|CA^{-\alpha-\varepsilon}A^\varepsilon x\bigr\| \geqslant \delta\bigl\|A^\varepsilon x\bigr\| \geqslant \delta'\|x\|.$$

That is, the invertibility condition on $CA^{-(1-\beta)}$ becomes more restrictive when β decreases. To admit more observation operators C one therefore seeks for values of β large enough. Combining both conditions forces to play with different values of β in different situations. In the following corollary we choose $\beta = \frac{1}{2}$.

Corollary 5.3. *Let $-A$ be the generator of a semigroup of contractions $(T(t))_{t\geqslant 0}$ on a Hilbert space X. If A has dense range and $\|CA^{-\frac{1}{2}}x\| \geqslant \delta\|x\|$ for all $x \in \mathcal{D}(A)\cap\mathcal{R}(A)$, then*

$$m^2\|x\|^2 \leqslant \int_0^\infty \|CT(t)x\|_Y^2\,dt \tag{5.3}$$

for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. In addition, if either A is invertible or C is infinite-time admissible for A then C is infinite-time exactly observable for A.

Notice that in view of Proposition 5.1, the hypothesis of a semigroup of contractions is necessity to be able to conclude in the case that C is admissible.

Proof. Since A has dense range and X is reflexive, A is actually injective (cf. [3, Theorem. 3.8]). By Theorem 5.2 A has a lower square function estimate. We then apply Theorem 4.2. Note that if C is infinite-time exactly observable for A we can approximate $x \in \mathcal{D}(A)$ by the sequence

$$x_n := n(n+A)^{-1}x - n^{-1}\left(n^{-1}+A\right)^{-1}x \in \mathcal{D}(A) \cap \mathcal{R}(A)$$

in $\mathcal{D}(A)$ (for the graph norm). Therefore, (5.3) extends to all $x \in \mathcal{D}(A)$. □

As in Corollary 4.4 we can obtain exact observability in finite time by adding a constant w to A.

Corollary 5.4. *Let $-A$ be the generator of a semigroup of contractions $(T(t))_{t\geqslant 0}$ on a Hilbert space X. Suppose in addition that $(T(t))_{t\geqslant 0}$ is exponentially stable. Let $C : \mathcal{D}(A) \to Y$ be bounded and suppose that there exists $\tau_0 > 0$ such that*

$$\left\|CA^{-\frac{1}{2}}x\right\|_Y \geqslant \delta\left\|T(\tau_0)x\right\| \tag{5.4}$$

for all $x \in \mathcal{D}(A)$. Then (A, C) is final state observable.

Proof. This is an immediate corollary of Theorems 4.6 and 5.2. Note that A is invertible since $(T(t))_{t\geqslant 0}$ is exponentially stable and hence $\mathcal{D}(A) \cap \mathcal{R}(A) = \mathcal{D}(A)$. □

Example 5.5.

(a) Let A be the uniformly elliptic operator

$$A = -\sum_{j,k=1}^{d} \frac{\partial}{\partial x_k}\left(a_{jk}\frac{\partial}{\partial x_j}\right)$$

with bounded measurable coefficients $a_{jk} \in L^\infty(\mathbb{R}^d)$. The operator A is defined by sesquilinear form techniques (see for example [22]) and note that A is not necessarily self-adjoint.
It is a standard fact that $-A$ generates a contraction semigroup on $L^2(\mathbb{R}^d)$. Consider the problem

$$\begin{cases} \frac{\partial u}{\partial t} = -Au, \\ y(t) = Cu(t) = \nabla u(t) \end{cases}$$

with $X = L^2(\mathbb{R}^d)$ and $Y = (L^2(\mathbb{R}^d)^d$. By the solution of the Kato's square root problem (see [1]), it is known that

$$\|\nabla u\|_2 \approx \|A^{\frac{1}{2}}\|u \quad \forall u \in W^{1,2}(\mathbb{R}^d). \tag{5.5}$$

On the other hand, the accretivity of A implies an H^∞ functional calculus on $L^2(\mathbb{R}^d)$. Hence it satisfies upper and lower square function estimates (4.4) and (4.5). In particular,

$$\|f\|_2^2 \approx \int_0^\infty \|A^{\frac{1}{2}} e^{-tA} f\|_2^2 \, dt. \tag{5.6}$$

This implies that $C = \nabla$ is both admissible and exactly observable for A in infinite time.
Note that the semigroup e^{-tA} is bounded holomorphic on some sector S_ω and $e^{-te^{iw}A}$ is a contraction on $L^2(\mathbb{R}^d)$ (see [22]). Taking the maximal angle w, we obtain a contraction semigroup $e^{-te^{iw}A}$ which is not holomorphic. Since $(e^{iw}A)^{\frac{1}{2}} = e^{iw/2}A^{\frac{1}{2}}$ we see that (5.5) holds with $e^{iw}A$ in place of A. By Corollary 5.3 we have

$$\delta\|f\|_2^2 \leqslant \int_0^\infty \|\nabla e^{-te^{iw}A} f\|_2^2 \, dt \tag{5.7}$$

for $f \in \mathcal{D}(A) \cap \mathcal{R}(A)$. We may consider the same problem on a bounded Lipschitz domain instead of $\mathbb{R}^d$. In this case, A is invertible. Hence $e^{iw}A$ is also invertible and we obtain (5.7) for all $f \in \mathcal{D}(A)$. This means that C is exactly observable for $e^{iw}A$.

(b) We consider A to be (minus) the Laplacian with Dirichlet boundary conditions on a bounded connected $C^{1,\alpha}$-domain Ω of $\mathbb{R}^d$. It is (the negative of) the associated operator with the symmetric form

$$\mathfrak{a}(u, v) := \int_\Omega \nabla u \nabla v \, dx, \qquad \mathcal{D}(\mathfrak{a}) = W_0^{1,2}(\Omega).$$

Denote by $p_t(x, y)$ the corresponding heat kernel. It is proved in [23] that

$$e^{\lambda_1} p_t(x, y) \leqslant \left[1 + K\left(\inf\{1, t\}\right)^{(d+2)/2} e^{-(\lambda_2-\lambda_1)t}\right]\rho(x)\rho(y)$$

where K is a positive constant, λ_1, λ_2 are respectively the first and second eigenvalues of A and ρ is the distance to the boundary of Ω. Denote by $(T(t))_{t\geqslant 0}$ the semigroup generated by $-A$ on $L^2(\Omega)$. It follows from this estimate that there exists a constant $K > 0$ such that for all $t > 0$ and all $f \in L^2(\Omega)$

$$\|T(t)f\|_2 \leqslant K e^{-t\lambda_1} \|\rho f\|_2. \tag{5.8}$$

For fixed $\tau > 0$ we have

$$\frac{\tau}{2}\left\|T(\tau)f\right\|^2 \leqslant \int_0^{\frac{\tau}{2}} \left\|T\left(\frac{\tau}{2}\right)T(s)f\right\|_2^2 ds \leqslant K^2 e^{-\tau\lambda_1} \int_0^{\tau} \left\|\rho T(s)f\right\|_2^2 ds.$$

In particular, if C is the multiplication operator $Cf = \rho f$, then the system (A, C) is final state observable. Notice that C is not boundedly invertible. Given now $\tilde{C}: \mathcal{D}(A) \to Y$ a bounded operator from $\mathcal{D}(A)$ into a Banach space Y such that $\|\tilde{C}A^{-\frac{1}{2}}f\|_Y \geqslant \delta\|\rho f\|_2$. It follows from the previous estimate and Corollary 5.4 that $(A, \tilde{C})$ is final state observable.

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